DOCUMENT RESUME

ED 029 781 24 SE 005 327

By-Machtinger, Doris D.
Experimental Course Report, Kindergarten, No. 2.
Syracuse Univ., N.Y.; Webster Coll., Websters Grove, Mo.
Spons Agency-National Science Foundation, Washington, D.C.; Office of Education (DHEW), Washington, D.C. Bureau of Research.
Report No-2
Bureau No-BR-5-1172
Pub Date Jun 65
Contract-OEC-6-10-183
Note-38p.
EDRS Price MF-$0.25 HC-$2.00
Descriptors-Arithmetic, Curriculum, *Elementary School Mathematics, Fundamental Concepts, Instruction, Kindergarten, *Learning, *Mathematics, Number Concepts
Identifiers-Madison Project

Reported are the results of an exploratory study that analyzed appropriate mathematical or "pre-mathematical" experiences for very young children of nursery school, kindergarten, or first-grade age. Among alternative approaches are the approach via sets, the approach via real numbers, and the approach via the physical act of counting. This study was based on the third approach, the physical act of counting. Included in the report is a discussion of learning, remarks about specific activities used with pupils, and a bibliography of related publications. (RP)

THE MADISON PROJECT

SYRACUSE UNIVERSITY
WEBSTER COLLEGE

OEC-6-10-183

EXPERIMENTAL COURSE REPORT / KINDERGARTEN

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Published by:

The Madison Project

Webster College

Webster Groves, Missouri 63119

TABLE OF CONTENTS

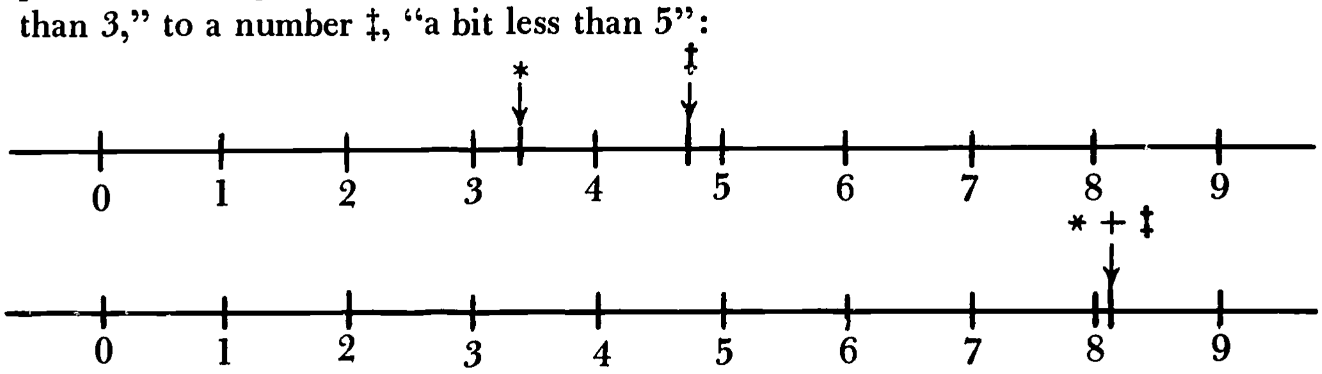
Madison Project "Experimental Course Reports" are intended to communicate to the profession —mathematicians, teachers, authors, publishers, and so on—at the earliest possible moment on the Project's exploratory curriculum work. By reporting as early as possible the Project can invite general professional participation and criticism in the planned evolution of school mathematics. At the same time, early reporting means, of course, that what is contained herein is a record of successes, guesses, uncertainties, and mistakes. The advantage of accumulating hindsight will surely dictate revisions in the curriculum suggestions which are presented in this report.

Emotional behavior of babies

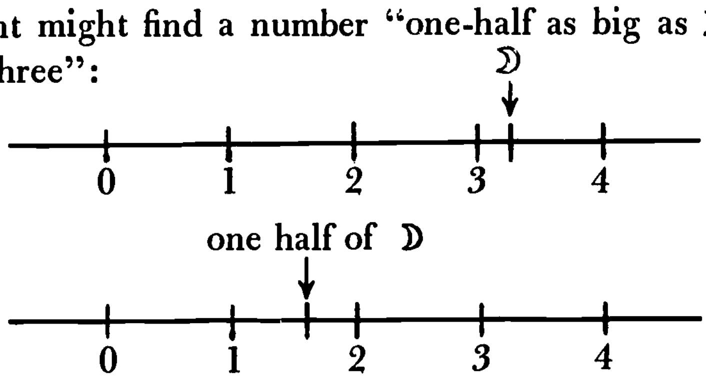
For several years, the Madison Project has been exploring appropriate mathematical or "pre-mathematical" experiences for very young children of nursery school, kindergarten, or first-grade age (that is, from 3 years old to perhaps 7 years old). For the past two years, three approaches have been in progress: one under the direction of Mrs. Doris Diamant Machtinger, a second under the direction of Beryl S. Cochran, and a third under the direction of Joan O'Connell. The present report is concerned with only the first of these, and only a part of that: the work done by Doris Machtinger during the academic year 1963-64. Further reports on the other efforts are now in preparation.

We are, by now, firmly convinced that this age group is the most elusive of any and is rivaled only by junior high school in the difficulty of developing suitable mathematical experiences. Indeed, many diverse approaches have been suggested, and comparing them is an extremely difficult matter. Our present emphases are tentative in the extreme.

Among alternative approaches, it is worth comparing three: the approach via *sets,* the approach via *real numbers,* and the approach via *the physical act of counting.* The first approach is well illustrated by the work of Professor Patrick Suppes;[1] the second was suggested by Professor Andrew Gleason, of the Harvard University Mathematics Department, and makes use of the number line, assuming from the very first that *every* point on the number line corresponds to a number. By using pieces of string, or otherwise, the student can add a number *, "a bit more than 3," to a number ‡, "a bit less than 5":



Similarly, a student might find a number "one-half as big as Ɗ," where Ɗ is "a bit more than three":



This is the most profound and subtle approach yet suggested. There would be, at first, no special restriction to whole numbers, nor even to whole numbers and fractions. Irrational numbers would be allowed from the outset.[2]

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1 Patrick Suppes, *Sets and Numbers, Book K-1,* Blaisdell Publishing Co., New York, 1962.

2 For those familiar with Cuisenaire rods, we could describe the Gleason approach by saying that it replaces the nicely cut Cuisenaire rods (which come in lengths of 1 centimeter, 2 centimeters, 3 centimeters, and so on, up to 10 centimeters) by a selection of twigs picked up at random in the school yard. Indeed, Professor Gleason has used his approach with children by just this method: the comparison, addition, subtraction, etc., of twig-lengths, which are, of course, entirely irregular.

The Madison Project has not, thus far, attempted any serious experiment with the Gleason approach. A somewhat similar approach has, however, been used by Professor Paul Rosenbloom of the MINNEMAST Project.[1]

Mrs. Machtinger has based her work on the third approach: the physical act of counting. There are many reasons for this choice; for one thing, children count whether you want them to or not, so it seems that discretion may be the better part of valor. Then, too, the Project believes strongly in *autonomous decision procedures:*[2] at every stage the child should have his own methods for testing the truth or falsity of statements, by himself, without the need to "take the teacher's word for it." For the young child, *counting* is an excellent "autonomous decision procedure." Indeed, if one does *not* develop the child's skill in physical counting, the child may well prove to himself that three plus four do *not* make seven, or that they only *sometimes* make seven—for children do not all count accurately, by any means. What happens to the docile child who is told by the teacher that three plus four make seven, but who can prove by his own experience that this is not so?

However that may be, and whichever approach one uses, we feel confident of one thing: to explore and to discover and to learn and to try things out is as natural to the young child as eating and breathing. This begins at least as early as the eighth or ninth month of the child's life—and by the time he reaches grade seven he may have given it up.[3]

This is a severe indictment of the way our culture teaches children to use their intelligence. They will need it all life long and should not retire it from service so prematurely.

We, consequently, feel that whatever approach is used, *the school must build on the child's natural exploration and natural growth.* This is not an easy task for the teacher, nor for the curriculum designer, but it is the only secure foundation upon which we can build. If most school programs neglect natural modes of learning and attempt to replace them by highly artificial modes, that is one measure of how large is the task which looms ahead of us all, teacher and curriculum planner alike.[4]

ROBERT B. DAVIS
Director,
The Madison Project

Webster Groves, Missouri
June 7, 1965

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1 Paul C. Rosenbloom, Minnesota Mathematics and Science Teaching Project, Minnesota School Mathematics and Science Center, University of Minnesota, Minneapolis, Minnesota 55455. (Professor Rosenbloom has, since this writing, moved to Teachers College, Columbia University, New York City.)

2 Robert B. Davis, "The Madison Project's Approach to a Theory of Instruction," *Journal of Research in Science Teaching,* Vol. 2, 1964, pp. 146-162.

3 Cf. especially: Eugene S. Wilson, "Amherst Dean of Admissions Seeks Questing Quotient (QQ) Instead of IQ," *Insight* (Science Research Associates), Vol. 3, No. 2, Spring, 1963, p. 1 ff; Mary Everest Boole, *The Preparation of the Child for Science,* Oxford, 1904; Paul Goodman, *Compulsory Mis-education,* Horizon Press, New York, 1964, pp. 53, 54, and p. 81; Z.P. Dienes, *The Power of Mathematics,* Hutchinson Educational Limited, London, 1964, especially pp. 17-25. The work of Professor Bruner on "action, imagery, and language" may also be suggestive here. Cf. "The Cause of Cognitive Growth," *American Psychologist,* Vol. 19, No. 1, 1964, pp. 1-15.

4 Among other kindergarten mathematics materials that deserve consideration are those developed by F. Mary Mason and Louise R. Smoluchowski at Miss Mason's School, Princeton, New Jersey, and those developed by Mrs. Joy Levi, also of Princeton, New Jersey.

DORIS D. MACHTINGER

EXPERIMENTAL COURSE REPORT NO. 2 / KINDERGARTEN[1]

SOME REMARKS ON LEARNING

By the time most children reach the age of five and start kindergarten, they have mastered the basics of one of the most difficult and abstract of tasks: they have learned to communicate in a language and are broadening their horizons every day, ever demonstrating their extensive abstracting powers and their insatiable thirst for knowledge.

Such abstracting powers and thirst certainly cannot be limited only to language, but must underlie a child's whole interpretation of the world—at least until he is taught otherwise. If these powers exist, and evidence strongly indicates that they do,[2] then this is the ideal time to introduce children to the basic concept of mathematics as well as language. Such abstracting powers should be preserved and cultivated. They form the foundation on which children can build their future mathematical encounters.

Mathematical experiences on all levels, but perhaps in particular on the primary level, must, as far as possible, have these four characteristics:

They must be intrinsically rewarding.
They must follow a natural learning paradigm.
They must offer an autonomous decision-making procedure.
They must have content which will be of future value to the students.

[1] The work described here was supported by the Division of Educational Research of the U. S. Office of Education; production of relevant films was financed by the National Science Foundation.

[2] Jerome S. Bruner, *The Process of Education,* Harvard University Press, 1963, p. 12.

These four characteristics are basic and will be discussed (and hopefully, clarified) in the following pages.

We shall say that a task is "intrinsically rewarding" if it is the kind of thing that a child is eager to undertake for its own sake, irrespective of "external" rewards or consequences. For many adults, completing a crossword puzzle may be "intrinsically rewarding"; an even better example might be learning a demanding hobby, such as wood-working or playing the bassoon.

Intrinsic Motivation

The material that children use in their early contacts with mathematics must be intrinsically motivating because, at such an early age, they are in an especially vulnerable position to form permanent dislikes for all mathematical activities based on a few bad experiences. We cannot force students to enjoy learning, nor can we buy their sustained interest with grades or extra recesses. True interest can arise only if the material, by its very nature, offers the students the motivation and reward necessary to sustain their attention. [1, 2] Whether a topic is intrinsically motivating or not can be ascertained only by observing student reaction to it.

The Psychology of Jean Piaget—A Learning Paradigm

Jean Piaget,[3] the Swiss psychologist, has spent considerable time observing children in learning situations and has described what he believes to be the procedure by which children learn. His description is as follows: the child observes an event and then forms a mental image to enable him to comprehend a simplified form of the event. As the child observes the event on subsequent occasions, he revises his mental image to encompass previously unobserved and now unaccountable facts. The learning process is a continuation of observations and subsequent revisions of the image, which each time renders the image more adequate for the latest observation and present use.

[1] Ibid., p. 14.

[2] Ronald Gross, "Reforms and Innovations in Education to Stimulate Learning in Lower Grades" (results of recent findings on development of intelligence), *New York Times*, September 6, 1964, Sec. 6, pp. 1-10.

[3] John H. Flavell, *The Developmental Psychology of Jean Piaget*, Van Nostrand Co., Inc., Princeton, New Jersey, 1963, pp. 41-84.

These revisions involve the dual operations of continually categorizing the image so that it fits into a pattern recognizable to the child (Piaget calls this "assimilation") and of adjusting the mental outlook to accept this flow of recategorizations (Piaget calls this "accommodation").

When a child perceives a tree, some green and brown light waves impinge upon the retina and send a meaningless image to the brain. The mind gives the image meaning by assimilating this information, categorizing it as vegetation, then further assimilating it by further categorizing it as a tree. The child accommodates by accepting and acting toward the image according to the new categorization.

When the tree begins to move, the new information is integrated and the image is again assimilated by now categorizing it as "Freddie with his camouflage suit". The child again accommodates by changing his mental outlook toward the image and now behaving as toward a friend rather than a tree.

Piaget further feels that each child must go through these stages himself, and that material presented to him which is too foreign to his present mental image cannot be integrated, but will appear unrelated or confusing.[1]

To follow Piaget's paradigm for child learning, we must offer children experiences that lend themselves to the formation of mental pictures which are easily assimilated. We must also watch the children for cues so that we can best help them extend their mental imagery, which, incidentally, will often be different from the teacher's.[2]

Autonomous Decision-Making Procedures

One of the most beautiful and captivating things about mathematics is that the truth value of an answer to a question in a given system is entirely dependent upon the system and upon logic, and independent of outside authority. "2 + 4 = 6" is true because of the definition of the numbers and operations—not because Miss Hagamen, the first grade teacher, said so. The product of zero and

[1] R. E. Ripple and V. N. Rockcastle (editors), *Piaget Rediscovered* (a report on Cognitive Studies and Curriculum Development), March, 1964, pp. 7-20.

[2] Bruner, op. cit., p. 25.

any other number is zero because of definitions, operations and logic—not because the third grade text book so decreed.

Children can be brought to realize these things for themselves if they have a checking device at their fingertips, and if they realize that the material contains an autonomous decision-making procedure independent of the authority of the teacher. They must realize that the very nature of the material is such that the facts themselves can be utilized to form an argument to support a correct answer and disprove a wrong one.

Future Value of Content

That the content of the material must have value for the students in the future is self-evident. At best, any material presented to a student offers him experiences that will not only help him explain what he is seeing in front of him, but will help him interpret all subsequent experiences.

KINDERGARTEN AT HILLTOP

The kindergarten material presented in the following report was developed during the school year of 1963-1964 at Hilltop School in Ladue, Missouri. The principal of Hilltop is Mr. Gerald Baughman. The kindergarten teacher, whose help in the development of this material was invaluable, was Mrs. Marilyn Ward. The material consists of a collection of topics that seem to offer children profitable, pleasurable experiences on which future mathematical learning can be based.

Counting Tubes

It is clearly apparent that children of kindergarten age have no difficulty reciting their numbers by rote: 1, 2, 3, 4, ... When, however, they count "things," they do not necessarily associate one, and only one, number to a "thing," nor do they necessarily have a "thing" for each number they utter. Further, the only way that they have of finding out if they have counted correctly is to ask the teacher and rely solely on her judgment.

Counting tubes were developed to offer children a checking device for counting. They are clear tubes, sealed at the bottom, and have a graduated scale marked off on one side. Each tube comes with some blocks which are the same height as the units of the scale on the tube. The children can count blocks into the tubes with the scales turned away from them, decide how many blocks they have put in, turn the tube around and read the number corresponding to the top of the column of blocks to check their answer.

The following are excerpts from reports of the class sessions in which the counting tubes were used.

November 10, 1963

I worked with the group as a whole. Each of the children received a counting tube with ten blocks inside. I started off by asking them to recognize the fact that there were numbers on the tube. We then opened them up and started counting the blocks. I asked them to put in seven, then three. I asked them how they knew that they had three blocks in the tube. Most of them said they could count and proceeded to do so. Marilyn said she knew because she "thinked." Jeffrey said he could tell because the edge of the block came up to the marking for three. This is the answer I was seeking. I then asked them how many I would have if I put in one more. They all got the correct answer. After we had four in the tube, I asked them how many we would have in the tube if I added another, etc. They could all add one to whatever number they already had. I asked them how to figure out three plus two. Most of them put three blocks in the tube, then two. Karen counted on the outside of the tube—first three, and then an additional two—without adding the blocks. She got the answer faster than anyone else. Actually, she used the concept of a number line. Most of the class did not get the result accurately. When we did it together, we had general agreement. They were intrigued with doing arithmetic. Even those who couldn't get the correct answer had fun. I feel I moved in a direction which involved the concept of both addition and counting.

November 12, 1963

I handed out the counting tubes, and, after we had emptied the blocks out of them, I asked the class what they meant by "five." Some of them arranged the blocks in a tower, some used the

tube, and some made a pyramid. I showed examples of the various ways of arranging the blocks and asked them which was correct and how we could tell. They decided the best way would be to count them together, which we did. I had three students then come to the front and count the number of girls who were sitting there. They came up with the answers, "ten," "ten," and "nine." I wanted them to suggest a way they could use the tubes to check the answers. One person said that we could count to ten in the tubes. (I don't think he had the idea.) I suggested that each girl drop one block into the tube. We did this and counted ten girls. We used the same procedure for counting the number of lights on the ceiling and the number of pictures on the wall. By the time we had counted the number of pictures on the wall, using the method of dropping a block into the tube for each picture, I think they were all with me. They all knew what I was doing.

I then went through adding "n + 1" and "n + 2" by asking the children questions such as: "If I have six blocks in the tube and I add two more, how many will I have?" They do remarkably well in handling the tubes on this type of question. Also, I asked them questions such as: "If I had eight blocks in the tube, how many would I have to take out to make six?" They were able to handle this type of question well, also.

Numerals to the Base Ten

Needless to say, kindergarten children have no conception of the place values inherent in our numeration system. An attempt was made to get at the major concepts involved by taking what might be described as an historical approach, or perhaps, as embracing the concept that ontogeny should recapitulate phylogeny.

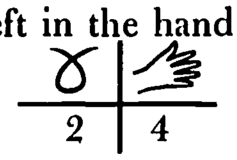
The first situations presented to the students were designed to bring out the concept of a "one to one correspondence." We proceeded by matching one "thing" at a time from one group of items with one "thing" from another group of items until all of the "things" in one group were used up. By using this process we can decide whether or not the two sets contain the same number of items; if they do not, this procedure shows which set contains more.

This concept is as ancient as the question, "How many?" Most likely, the ancient shepherd kept track of the number of sheep he

possessed by keeping a pebble in a sack for each of his sheep. If he wanted to know who had more sheep, he or his neighbor, he lined up his pebbles and matched each one with one from his neighbor's sack until one of them ran out of stones. The students had similar experiences, without the sheep of course.

At first, the task was simple. Then the teacher began to introduce successive complications, until any attempt to use the direct matching procedure finally became impossible. The students had to devise some new method to circumvent the difficulties. They did so by developing sounds for each cardinal number—in other words, counting. They made up symbols for each natural number through twelve. After deciding that to make up an entirely unique symbol for each number was nonsense, they devised a rule which said that you can hold as many as nine stones in your hand, but when you try to hold ten, it is too much. At this point you must put them all in a bag, close the bag, and start over until you need another bag.

They kept track of the bags and the stones left in the hand on a chart. For example, twenty-four stones would be two bags of ten stones and four stones left in the hand. On a chart this would be



This procedure will be made clearer by reading the class reports pertinent to the topic.[1]

November 19, 1963

> Today's lesson was, by far, the most successful I have had so far. Last week we noticed that the children were having trouble reading numbers above eleven. Marilyn Ward did some exploring with them on Friday and Monday and discovered that they had no understanding of the reason for using two digits for numbers above ten to a hundred. They could understand having one shape for the number of elements in a certain group when the number is less than nine, but why use two shapes for numbers above nine? Why not just have another shape? This problem gives direction to the classes for the next few sessions. We are developing the numerals to the base ten.
>
> For today's lesson, we cranked up our thinking machines and attacked the problem that follows: Two helpers each have an envelope; I put "n" rocks in one envelope and "m" rocks in the other. The problem is: Who has more rocks?

[1] This material is available on the film listed as No. 79 in *U.S.O.E., 1965.* (Cf. p. 36 of the present volume.)

The first inclination was to count the rocks. However, I told them that we were going to do something odd: we were going to forget that we knew numbers. Jeff pointed out that he knew them, and he couldn't forget them, so I told them to pretend that they didn't know their numbers. Well, back to the problem. Some of them resorted to blind guessing. Finally, someone came up with the idea of just looking at the rocks—maybe we could tell from that. We looked, and they could tell. I mixed up the rocks and meted them out again. This time we again wanted to solve the same problem, but differently.

The new solution was to make rows and see which row was longer (this was Jeff's idea). This was a very good solution, and I was pleased with it. Then I made the problem harder. One person stood on one side of the room and one on the other side. They each got envelopes with pebbles. Now we had the same problem, but we couldn't make rows because we couldn't cross the room. The solution to the problem was to hit the rocks against the board, one at a time, alternately, and see who ran out first. The only complication was seeing whose turn it was, but this didn't present any trouble. The children were fascinated by this. We must have done this eight times before I went on to a new problem.

The new problem was with feathers (I got some feathers, which, luckily, were on hand for a unit on Indians). We again had the same problem, however, it was complicated by the fact that you can't hear feathers click against the board. How should we solve this one? Jeff suggested that the helpers drop the feathers on the floor, alternately, so that we could see who ran out first. We did this.

Next, I really complicated the problem. What if two people had feathers on opposite sides of a dark room? Now, how could we tell who had more? This was a real stumper. They couldn't think of anything else but counting. Well, fine! Now I had established a need for counting. As a matter of fact, counting, they realized, would have helped us all along. We could have used numbers for all other problems if we hadn't pretended we didn't know them.

Up to this point, I am almost sure that everyone was with me.

With the need for numbers established, I then held up one feather and asked them what we should call "this many" things. "One," they chimed in. So we did. Furthermore, we

decided that three objects were "3" and four objects were "4." I avoided taking the numbers in order because I am working strictly with cardinal numbers. I'll get to order later. I shall continue to avoid order until it is no longer practical to do so. Then I'll let them discover the natural order of the positive integers. After we decided on the fact that one feather was "1," I added two more and asked, "How many do we have now? What number should we call this amount?" Terry said "two" because "one" was already counted; so I put the one feather aside and picked up another in its place, and she happily agreed to call the new set "three." We shall move from this stage to grouping base ten, but that will take a few lessons.

November 21, 1963

Today's lesson, as I saw it, was a key one in the understanding of the base ten number system. We started with a review of how to find out who has more objects, if we can't count. We ran through two examples, and then we decided that things would be much easier if we could count.

I tried to convey the idea that the words we say for numbers are means of communicating to someone else how many things we mean, and it works because everyone knows the code. To do this, I put various numbers and objects up on the board, and we decided what symbol we should use for each. Again, since I am concerned with the cardinal numbers, I did not put these numbers in order.

I let them use the conventional numbers through nine. At nine, I stopped and tried to convince them that as long as we knew what we meant by our symbols, we could make up any symbol we wanted. We made up our own symbols for ten, eleven, and twelve. The symbols were Jeff's symbol (to mean "ten"), David Ornstein's (to mean "eleven"), and Karen's symbol (to mean "twelve").

As we developed each symbol, I had the children hold up the appropriate number of fingers. The enthusiasm on their faces as they were able to get the right number indicated to me that they were really involved. Martin was thrilled when he meticulously counted out ten fingers and found that he had just the right number. Terry Cohen counted out nine fingers and had the little finger left over. She couldn't make it stay down, so she decided that she couldn't hold out nine fingers.

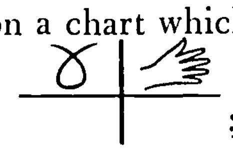
I think it is significant to point out that the children did not

copy the number of fingers their neighbors held up, but really counted for themselves.

We had quite a gallery of observers (Dr. Davis, Roy Hajek, Gordon Bennett, Mr. Baughman, and, of course, the classroom teacher, Marilyn Ward); but the children, except for a few who had been standing up to face the class, were totally unaware of their presence.

December 3, 1963

Today's session was good. We really got into grouping in tens. I used rocks and clear plastic bags. We made a rule that we would count out our objects, and as soon as we had as many objects as we had fingers, we would put them into a bag. We kept count of how many bags we had and how many rocks were left in the hand on a chart which looked like this: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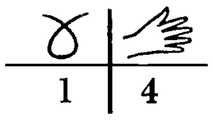
that is, fourteen would be one bag and four rocks left in the hand, or



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Five students (out of about eighteen) seemed to clearly understand what we were doing. They were Karen, Barbara, David Kohm, David Ornstein, and Jeff. Most of the rest of the class seemed to be able to get the correct answer to the questions I asked, but I suspect that they didn't have any insight into what I was doing.

In today's lesson, I presented the material in as many different ways as I could. I started out with a pile of rocks, broke it into "bags" and "extras," and drew the chart to tell how many rocks there were. Then, we started with a number and told how many "bags" and "extras" it represented and drew the chart. Next, we started out with a charted number and told how many "bags" and "extras" it represented and what number that was. I covered every sensible variation I could think of.

December 5, 1963

In today's session, we again worked with grouping "tens." This time we used bouquets (ten flowers to a bouquet) and single flowers, instead of rocks and plastic bags. I don't suggest this, because a handful of, say, seven or eight single flowers looks very much like a bouquet of ten and can cause confusion.

I did one thing that worked very well. I divided the children

The children in the experimental kindergarten classes have made several films showing actual classroom lessons. These films were made at KETC-TV, St. Louis, Missouri. The teacher is Doris Machtinger. For information about these films write to the Madison Project.

into four groups of five or six and gave each group a handful of flowers. The group's task was to find out how many flowers they had by finding how many bouquets of ten and how many extras. Leigh's group was particularly interesting. Leigh really took charge. He delegated Karen to write, and he personally counted out bouquets of ten and handed them out until he had no more flowers. He then asked his group who had bouquets and who had extras and counted them. Then he directed Karen to write the answer on the chart.

I felt that there was no use continuing with this unit any longer. Some of the students understood what we were doing; the rest will have to mature a little more before reaching this understanding. I'll go back to this material in a few weeks.

It may be well to discuss briefly the matter of verbalization and level of expected achievement and performance. Most of the children in the class cited above really were able to handle the material in this unit. The video tape on this topic[1] shows the children working in small groups[2] and busily, noisily constructing charts, which indicates that most of them understood the method used to describe the place value concept, and, also, that a few did not. Probably none of them would have been able to sit down and explain precisely what they were doing. A few could have said that fifty-nine is five "tens" and nine left over, but the unit was not designed to get them to this verbal level of achievement; it was designed to help them build an intuitive feeling for the system and to give them some of the necessary experience for preverbal concepts in this area.

With most of the topics covered, the typical lesson was designed not to produce accuracy in verbal explanation nor in computation, but rather to provide experiences in thinking about properties that are usually taken for granted, to provide experience in handling and developing useful concepts as tools to fill a need (for example, the concept of place value lessens the number of symbols to be learned), and to provide experiences in processes which have built into them autonomous decision-making procedures.

[1] This material is available on the film listed as No. 79 in U.S.O.E., 1965. (Cf. p. 36 of the present volume.)

[2] The film clearly shows four groups of children working on a given problem with minimal supervision, and producing some impressive results. This film consequently contradicts the erroneous but commonly-held assumption that children of kindergarten age cannot work effectively in small groups without constant adult supervision.

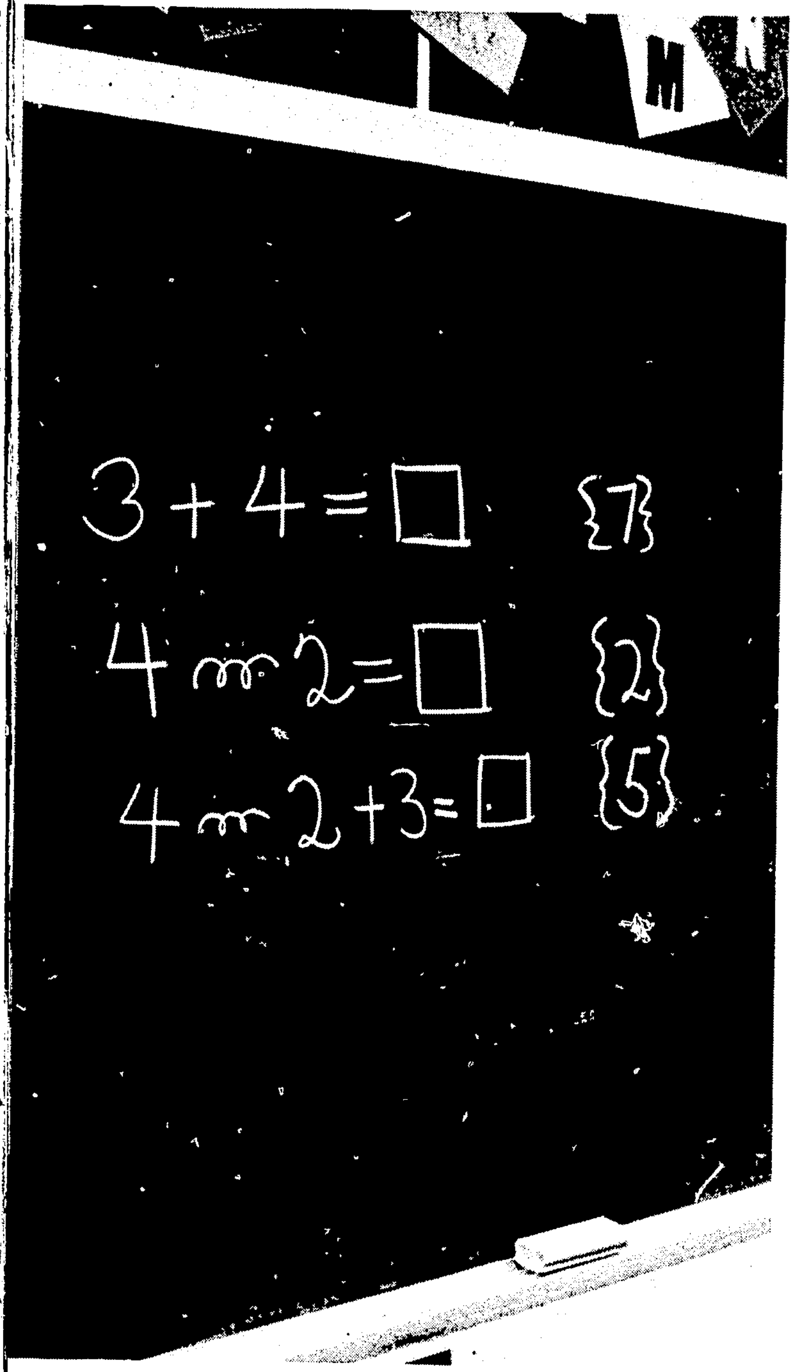
The fact that the students occasionally write their numbers backwards or upside down, or forget to write a plus sign, or use two parallel vertical lines instead of horizontal ones for an equal sign, is not particularly important at this stage and usually is not emphasized any more than by saying, "This number is backwards, but that's all right—we all know what he means."

Ordering

The letters of the alphabet are memorized in a specific order. The counting numbers are also memorized in a specific order. But that is as far as the analogy goes. The arrangement of the letters of the alphabet is purely arbitrary, but the counting numbers have a natural order to them which is quite within the abstracting power of kindergarten students.

December 18, 1963

The first thing we did was to order the positive integers. We did this in the following way. I put a "one" up on the board and asked, "What comes next?" They told me, "Two," then "Three," then "Four." At this point, I stopped and asked why they put them in this order. To stimulate the proper line of thinking (after getting answers such as, "That's what comes next," or "Because it's right," or "Cause you're supposed to"), I called up Martin, the tallest boy in the room, and Leslie, the shortest girl. I asked the class the question, "What if I had to put one of the students in front of the other, and I wanted them both to be able to see? Who would I put in front?" Most of them yelled out "Leslie"; a few called out "Martin." I placed Martin in front of Leslie and asked Leslie what she saw. She told me, "Martin's backside." We then put Leslie in front, and both could see. I asked them why we put Leslie in front, and a few answered, "Because she is smaller." I then asked, "Why do we put one in front of two, two in front of three, etc.?" They told me, "Because the first number is smaller." I asked them if it is true that we are always correct if we put a larger number after a smaller one and they said, "Yes." I put five after four, and they agreed; then I put eight after five, and they objected violently. So I showed them that eight is, indeed,



larger than five, and asked, "Why not?" Kim Berg beautifully verbalized the fact that all the other numbers differed by one, but this combination didn't. (Interestingly, they knew they differed by three). So we used Kim's statement and filled in the six and seven. Also, we did a little abstracting. I asked them what number would come after "giggly pooh." They told me, "Giggly pooh and one"; after "Martin-Michael" they told me, "Martin-Michael and one"; et cetera. We then played a game, girls against boys (just for five minutes), in which I put down a number, and they were to give me the next number. They did well.

The concept of one number being "larger" or "smaller" than another is often confusing to kindergarten students. An inevitable question that must come up is, "Which number is larger, one or two?"

December 19, 1963

Today's lesson was, to say the least, different, because Mort Schindel was walking around sizing up the class for a filming. I didn't want to go into any new material with the children just before the filming, so we went over "ordering." We reviewed "Kim's rule," that "in the positive integers each number is only one higher than the one before it." We spent some time straightening out the difficulty that arises between the size of a number and the size of the picture of a numeral. I did this by first asking, "Which number is the bigger of two numbers?" Next, I drew a big picture of the small number and a small picture of the big number and asked, "Which picture is smaller? Which number is bigger?" Then I had various children come up to the board and draw big pictures of small numbers and small pictures of big numbers.

Open Sentences

An important concept in a modern pursuit of mathematics on the elementary level is the distinction between "true," "false," and "open" sentences. The meaning of "true" and "false" sentences is reasonably self-evident. An "open" sentence is a sentence which, as it stands, is missing a "bit" of information; when supplied, this information will make the sentence either true or false. (The

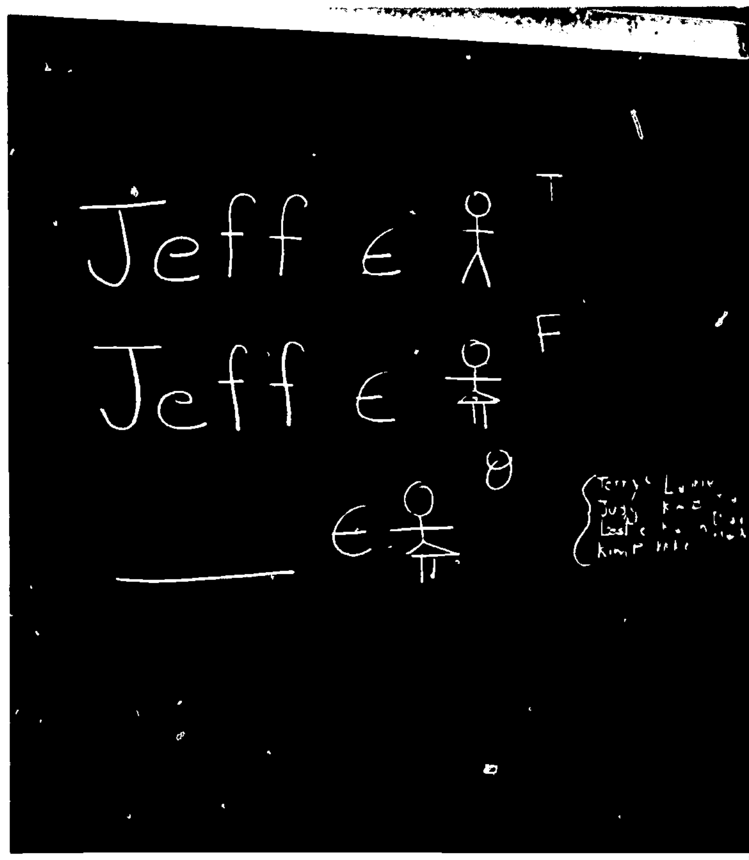
place where the information is missing is usually indicated by some form of a "variable.")

Traditionally, the variable in a mathematical open sentence (the unknown in an equation) has been represented by an x, y, z, α, β, γ, or other such symbol. The use of □s, △s and other such polygons for variables has become popular because the task involved is described much more clearly than if expressed by letters. The task described by the statement $\square + 3 = 5$ is much more implicit than that described by $x + 3 = 5$. In fact, if you do not utter a sound, but walk into a class and write $\square + 3 = 5$ on the board, students who know their addition facts will raise their hands and say "2"; whereas, if you write $x + 3 = 5$, you may draw only blank stares .[1]

January 14, 1964

Today I tried to introduce the concept of true sentences, false sentences, and open sentences. I also introduced the concept of truth sets. This lesson was very exciting for both me and the students. I wish that we could have had it on film, or at least somehow recorded. I started out by asking the children if they knew the letters, T, F, and O. I felt that it was important to make sure that they knew these letters so that we could indicate whether the sentence we were dealing with was true, false, or open. Then I asked them if there was anything wrong with the following statement: "Michael is a boy." They agreed that there was nothing wrong with that sentence. Then I asked them if there was anything wrong with this sentence: "Kim is a boy." They all agreed that the sentence was not true. So I asked them if we could say it was false. They said, "Yes," and all knew the word "false." Then I introduced symbols. We must remember that this is a kindergarten class and they cannot read; this makes writing sentences a little bit difficult, but not impossible. I used a stick symbol of a boy for a boy, a stick symbol of a girl for a girl, and the symbol $\in$ to mean "is a". Then I wrote the sentence,

"Kenny $\in$."



They were all able to read it and were able to establish that it was true. We then put a "T" after it, and they agreed on the

[1] Cf. Robert B. Davis, *The Madison Project—A Brief Introduction to Materials and Activities,* the Madison Project, 1962 (revised 1965).

convention of putting a "T" after statements which are true. I next wrote the sentence,

"Kenny ∈  ."

They agreed that it was false, and told me to put "F" after it.

I did this. Next I wrote,

"—— ∈  ."

When I asked if it was true or false, Kim Berg said, "We don't know. We don't know what goes in the blank." I told the children at this point that this was correct, and such sentences are called "open." They agreed that we could put an "O" for open sentences.

Next we started discussing what could go in the blank to make the sentences true. I drew the truth set brackets on the side of the board, and they listed the children's names. Whenever they gave a name, I wrote the name in the blank. We then wrote the sentence and decided whether it was true or false. If the sentence was false, I listed the name underneath a sad face. This indicated that it did not make the sentence true. If the sentence was true, I put the name in the truth set. After we had four examples of names that didn't work, and three names that did work, I asked them what other names would make the sentences true. They told me, "Any boy's name will do."

After that, I wrote sentences involving numbers. I wrote, "$1 + 1 = 3$" and asked them if this was true or false. They immediately told me it was false and suggested I put an "F" after it, which I did. Next I wrote "$2 + \square = 3$" and asked if this was true or false. A couple of the children thought it was false, but most of them agreed that we did not know. So we called it an open sentence and designated this by putting an "O." We then investigated which numbers would make this correct. They all knew the answer was "1," so there was no discussion. After a few more sentences of this nature, I attempted to introduce the "greater than" and "less than" signs. I didn't emphasize this; I just brought it in as something else we would talk about as being true or false.

January 29, 1964

I opened today's lesson by reviewing the meaning of "T," "F," and "O." These, of course, mean "true," "false,"

and "open," referring to true sentences, false sentences, and open sentences. I wrote on the board,

"Judy $\in$  "

in the symbolism we had made up last time, and asked them what I should put on it. They told me I should put a "T" on it. I did so and then wrote,

"Martin $\in$  ,"

and they told me to put an "F" on it. When I wrote,

"—— $\in$  ,"

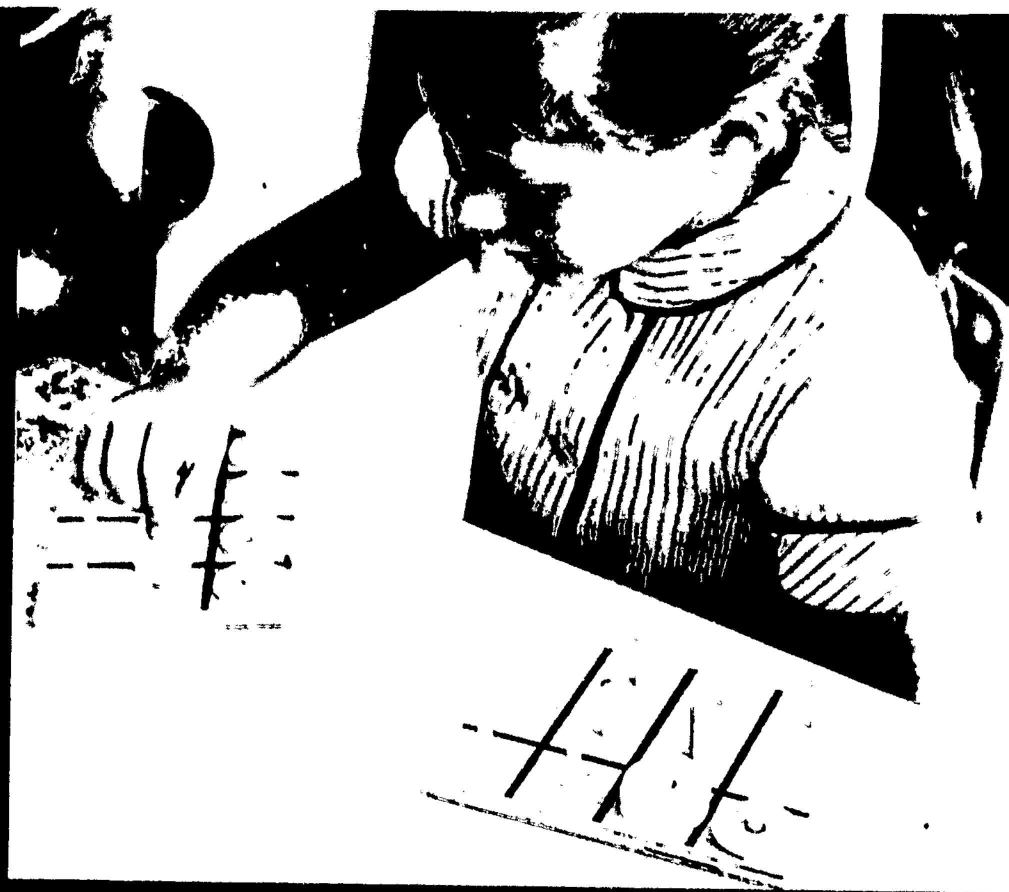
they remembered that this was an open sentence, and should have an "O" on it. I then drew the brackets for the truth set and asked if anybody remembered what it was called. David Kohm remembered it was called the truth set.

We then considered further the statement

"—— $\in$  "

and filled in various names which made the sentence either true or false. After finding three answers which made the statement true, I asked them how many things would go in the truth set if we put everything we could in it. There was a little discussion about whether the answer was eight or nine, because some of the children trying to count how many boys there were in the room forgot to count themselves; however, the general consensus, eventually, was that there were nine names to go in the truth set. We did the same in a sentence involving girls.

I then introduced, formally, the concept of "less than." I wrote down "1" and "2" and asked which was smaller. They decided that "1" was smaller, so I showed them the appropriate symbol to put between the two numbers. Then we decided that the statement that "$1 < 2$" was true and put a T on it. Next, I wrote "$0 < 4$," and they decided that also should get a T. I wrote down "$5 < 3$," and they immediately decided I should put an F on it. I asked them what I could do now to make this a true sentence. Michael came up to the board, took a piece of chalk, and turned the inequality sign around. This was an answer that I really hadn't expected, and it was a concept that I was not going to introduce. I think I will continue with the idea of keeping a smaller number on the left side, if for no other reason than its aiding reading readiness, besides the fact that on the number line you actually do find the numbers in this



order. Jeff came up to the board and wrote "$3 < 5$" as his way of correcting the statement. We all agreed to this and put a T on this, as well as Michael's solution.

The next sentence I wrote was "$\square < 4$." They decided that that was an open sentence, since we didn't know what should go in the box. We started looking for answers for the truth set. The only answers the children came up with were 1, 2, 3, and 0. I told them that there were other answers: numbers like $1\frac{1}{2}$, $2\frac{1}{2}$, $3\frac{3}{4}$, et cetera. Then I told them that there were still other answers for this truth set and asked if anybody knew what they were. (I was looking for negative numbers.) None of them knew, so I didn't press it any further. I then wrote the sentence, "$2 < \square < 5$," and they were able to find the truth set for this.

Numbers, Addition and Peg Boards

One of the major criticisms of the "traditional" methods of teaching addition and subtraction is that the number facts are dictated almost arbitrarily by the teacher. A child is taught that $3 + 4 = 7$ and told to memorize it, or told to visualize "three apples plus four apples—hence, picture seven apples." (A virtually impossible task. Just try it!) Children need experience in handling objects. To gain this practical experience, the students are given peg boards and washers, developed at Leicester, England, by Leonard Sealey and Z. P. Dienes.[1]

The peg boards are small boards with twelve pegs sticking up and an assortment of round washers in five colors which fit on the pegs. This supplies the children with something to manipulate to get their answers, and, also, gives the children concrete experience in handling quantities, before abstracting. The children must have concrete experience in adding three "things" plus four "things" before they can reach a level of abstraction that allows them to say, "I can add three washers plus four washers and get seven washers." Further, we can add three tomatoes and four tomatoes without going to the store to buy tomatoes, because

[1] Leonard Sealey is presently Principal of North Bucks College of Education, Bletchley, England; Z. P. Dienes is Reader in Psychology at the University of Adelaide in Australia.

it will be just like adding washers. As a matter of fact, we can add three knives plus four knives, even though we do not know what knives are, because this, too, will be just like adding washers. In fact, we can sum it up by saying that three of anything plus four of anything gives seven of them, or, simpler, three plus four is seven.

January 29, 1964

At this point in the lesson we went back to the table, and I handed out peg boards and ten washers to each child. My instructions to them were to do whatever they wanted to do with the peg boards, but that they should attempt to do something that had to do with numbers or mathematics. I observed the following play. Some of the children used them to pretend they were playing tic-tac-toe with themselves. There was general interest in counting the number of washers they had: first they counted them in their hands, then they put them on the peg board in one array and counted them that way, then they changed the array and counted them again. Some of the children performed addition problems of the following nature. They would put a certain number of washers, say four, on their peg boards, one washer on each peg. Then they would take in their hands two more washers. After they had put these two washers down on two different pegs, they would say, "4 + 2 is 6." Then they would look at that and take off three and say, "6 take away 3 leaves 3." They went through various combinations of things that add up to numbers less than ten, because there were only ten washers. One girl had a combination of the order 4 + 2 + 3 is 9.

They also worked on concepts of "less than" and "greater than." They did this by putting a certain number of washers on one peg and a smaller or larger number on the peg next to it.

Then they would say, "This peg has more washers on it than that peg," or vice versa.

They worked for twenty minutes with the washers and peg boards without any silliness; that is, without throwing the peg boards or the washers at anyone. They seemed very interested in them and very enthusiastic.

(The peg boards and washers, at this point, seem to have a great value, if in nothing else, in giving the children pre-mathematical experience in handling objects which have mathematical meaning. I will certainly use the peg boards and

washers again, because I feel that the children gained something by using them.)

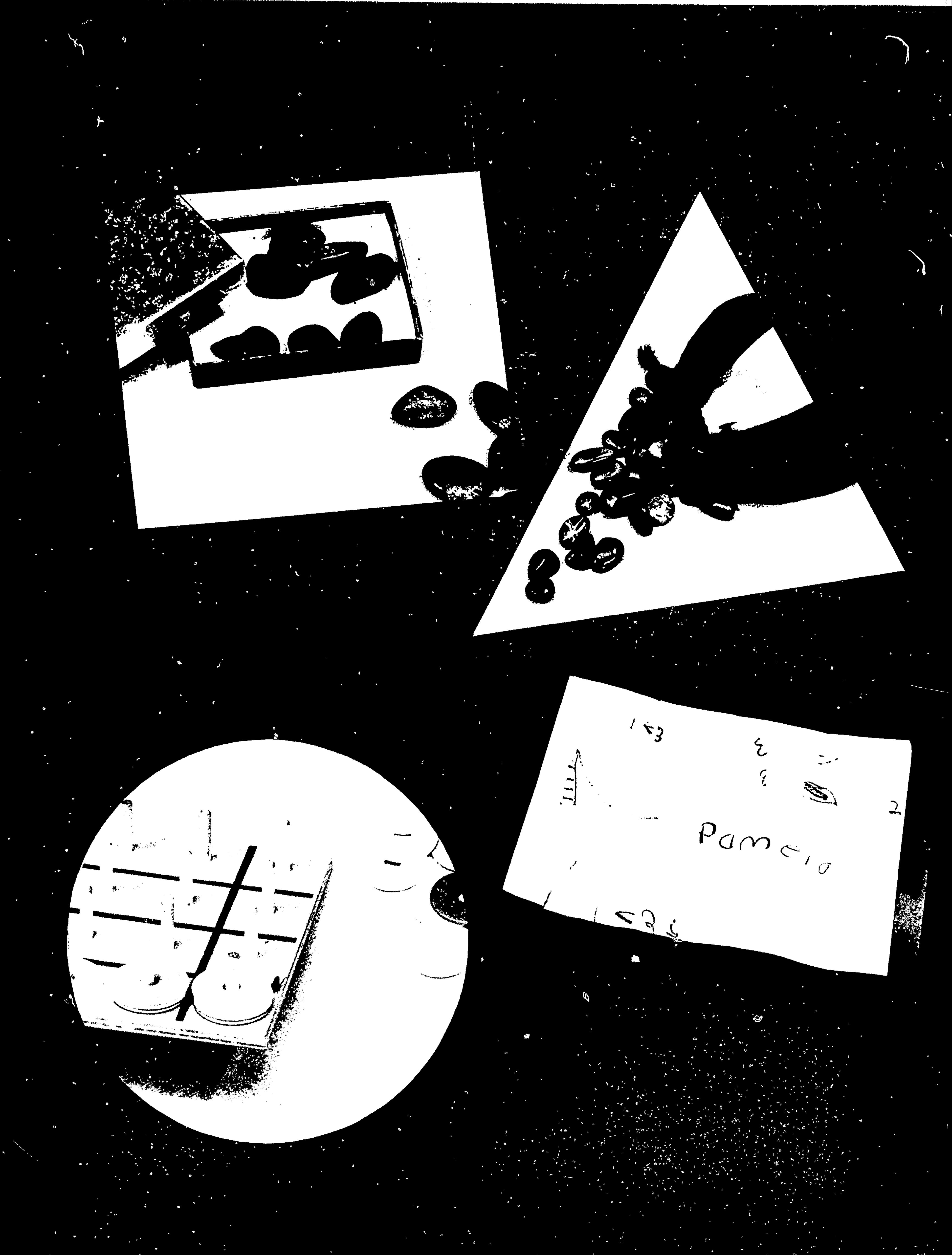
Next, I put to a vote whether they would prefer playing tic-tac-toe or playing with the washers. They voted eleven to seven to play with the washers. I handed out the washers and they played with them on their own for five minutes. Then Dr. Davis took over the class and asked the children to show him, with the washers and peg boards, what they thought he meant when he said $2 + 3$. He wrote $2 + 3$ on the board, and the children showed him. Most of them got it correct. Some of them stacked washers up one on top of the other; some of them put a row of two and a row of three; and interestingly, some put a row of four and a row of one. I must point out that the peg boards are 3 by 4 pegs long. Those who had a row of four and a row of one were able to clearly explain what they meant by it. They really did have the concept. Then Dr. Davis did the same for $3 + 4$, with similar results.

February 13, 1964

In today's lesson I had the children sit on the floor in front of the blackboard and handed out the peg boards and washers. I put a 3 up on the board and asked them what made 3. They told me, "2 and 1," so I asked them to show me $2 + 1$ with their boards. After we had done 2 and 1 and demonstrated that 1 and 2 is really the same thing only turned upside down or sideways or however the board happened to be arranged, we went on to a different combination, namely, $1 + 1 + 1$, and then $3 + 0$. We always looked at the boards to see how they had created the number 3. Needless to say, the children gave me the combinations. They determined them individually on the peg boards.

After we finished the 3, we went on to 4. We first did $1 + 1 + 1 + 1$, showing it on the peg boards. Next we did $2 + 2$ on the peg boards, and then $2 + (1 + 1)$. We showed that $2 + (1 + 1)$ is really the same thing as $2 + 2$, because the $1 + 1$ is equal to 2. And, of course, the next step was to commute the $(1 + 1)$ and the 2. Our next combination was $3 + 1$, then $1 + 3$, $4 + 0$, and $0 + 4$.

After we had done this board work, the children sat down in their seats. We handed out paper and crayons and then covered up the part of the board which had the combination adding up to 4 on it. They were told to write down a 3 and


Pamela

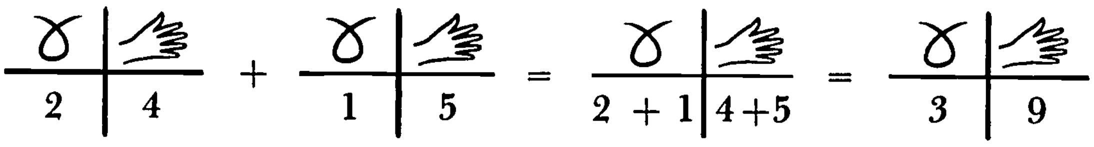
either figure out, or copy, those combinations which add up to 3, and that when they had finished, they could go on to the 4's. Fifteen minutes before the end of class, we uncovered the 4's, and those who were not able to figure them out copied them .

Addition: Stones and Charts

The novelty of the washer boards would wear off fast if we did not do addition with other media. A "game" the children loved was "How many stones in the box?" This involved putting two or more quantities of stones in a little golden box, closing the lid, and then deciding how many stones were in the box. The moment of truth, after all opinions were ventured, came when the box was opened and the stones counted.

The excitement generated by this "game" is unpredictably great. However, the method is not foolproof, especially when the children want to add very large numbers. They can easily count the stones into the box incorrectly.

To add large numbers we used charts, the same charts discussed above in the unit on base ten numeration. This broke the problem of adding 24 + 15 into the two smaller problems of adding 2 + 1 and 4 + 5. (Remember, kindergarten children do not know the "rules" on how to add in columns.) 24 + 15 became



Carrying, and later borrowing, came easily out of this system.

March 31, 1964

Today I decided to try a new way of addition because of Marilyn Ward's concern about children writing when some of them are clearly having coordination problems. I wanted to continue practicing addition, so I devised the following idea. I got a box and some pebbles. I called three assistants up to the front of the class. One of them held the box, and the other two put a certain number of stones into the box. As the first one put his stones into the box, he counted them, and I recorded it on the blackboard. I put a plus sign between the two numbers, an equals sign after them, and a box (☐) on the right hand side

of the equation. I then closed the box that the children had put the stones into and asked the children how many stones were in the box. This would give us the truth set for the equation that I had written on the blackboard. The first set of numbers that the children chose to use were 15 + 9.

I was a little worried about the children using such large numbers for their initial try, but the combination gave them no trouble at all, and Leigh immediately came up with the answer, "twenty-four."

The next set of numbers they chose was larger: 14 + 21. The first child counted fourteen stones into the box, and the next child began to count his into the box, starting with 1, 2, 3, ... I noticed, at this point, that when we were counting five (that is, when counting the fifth stone), one group of children, consisting of Leigh, Kim, Jeff, David Kohm, Karen, and a few others, were on nineteen; and when we were on six, they were on twenty, et cetera. By the time we had gotten up to twenty-one, they were on thirty-five and knew the answer.

There had been some slight error in dropping the stones into the box, and I knew that if they had counted the number of stones in the box, they would not have come up with the correct answer, not because of any fallacy in their method, but because of the inaccuracy of the child dropping the stones into the box.

I wanted to find another way of verifying that 14 + 21 was, indeed, 35. To do this, we wrote 14 and 21 as numbers on charts, as we had done previously. We then added the number of bags and the number of stones left over and came up easily with the answer that there were 35 stones in all.

Abstracting

How far can kindergarten children abstract? Surely, no one knows the answer to that. Here are some instances that came up in this particular class.

November 12, 1963

I ended the class with the questions: "What is the largest number in the world? What is the highest you can count?" Terry said that twenty-nine was the largest. The whole class

told us she was wrong. Leigh thought the largest number was a "zillion-willion." Karen came up with the beautiful concept, even if incorrect, that you just go on and on and on and on, until you start over again with one. (She traced a circle with her finger while she said this.) Jeff came up with the correct answer that there is no end—you just keep on counting.

March 31, 1964

I wanted to develop an accurate concept of the role zero plays in the real numbers. To do this, I said, "Tell me something about zero." At first, the children told me that zero is nothing. Kim Berg showed me the palm of her hand and told me that there were zero people in her hand and zero elephants in her hand. Karen then told me if you have zero pieces of a dime in one hand and zero pieces of a dime in another hand, and you put your two hands together, you still don't have any money. I told them that this was true and wrote on the blackboard, "$0 + 0 = 0$." Then they told me that $1 + 0 = 1$, and I wrote this on the blackboard. They did this with other numbers and with the word, "schnigelfritz" (that is "schnigelfritz" + 0 = "schnigelfritz"), then with Karen's symbol, ⊡ + 0 = ⊡. After this, David Kohm got up on his knees and announced, "If you have something and you add zero to it, you still have the thing you started out with." This statement was dubbed "David Kohm's rule" and will be referred to as such in the future. Karen also wanted to have her name immortalized, so she came up with the statement that if you have zero of anything, then you still don't have anything. This will go down in history as "Karen's rule." I feel that both of these rules are significant observations for kindergarten children. Indeed, they are both statements of true mathematical importance.

This lesson brought up the question of identities. (I wondered, later, if I should have introduced the identity, "$\square + 0 = \square$," today. I am considering doing it in the next lesson.)

April 2, 1964

We tape recorded today's class and will tape record the lessons from now on; so if anybody has any questions about this, or any subsequent lessons, they may ask to see the tapes. I started the class out by asking the children if they recalled "David Kohm's rule." They could not recall it, so we had to reconstruct it. I did this by asking them to search their memories, and in a

very few minutes we had brought it back. David remembered his rule—that if you take some number and add zero to it, you get the same number again. I asked them what would happen if we started out with the box and added zero to it. They told me it would equal □. I wrote "□ + 0 = □" on the blackboard, and I asked them if the statement was true or false. They told me it was true and, also, open.

The next thing that I wanted to do was to put "Kim's rule" in a more abstract form: namely, expressed in boxes. "Kim's rule," we will recall, is that if we have a number and we want to know what number follows it, it is that number plus one. I started out by asking them if they remembered "Kim's rule," and they did. I then put "5" on the board and asked what follows; they told me, "6." I asked them how else we could write 6 to show that it really did follow 5, and they told me, "5 + 1." I then asked what comes after 5 + 1, and they told me, "7." I asked how we could write 7 so that we knew it came after 6, and they told me, "6 + 1." I asked them how we could write 6 + 1 so that we really knew that it came after 6 (which was expressed as 5 + 1). They told me that we could express it as "5 + 1 + 1." I then put up a box, and asked what number would follow it. Kim told me that it would be followed by □ + 1, then, □ + 2. We also showed that □ + 2 could be written as □ + 1 + 1.

I next asked them if we could discuss some interesting things about addition. When I asked them for two numbers and their sums, I got a number like 5 + 6 = 11. Then I asked, "If we know that 5 + 6 = 11, what else can we find out?" They did not immediately catch on to what I was asking for: namely, that 6 + 5 was equal to 11. Eventually, somebody pointed out that 10 + 1 = 11; at the same time, somebody else was trying to point out that 1 + 10 was equal to 11. Kim verbalized this by saying that 10 + 1 and 1 + 10 are "just backwards." We then looked at 11 + 0 and 0 + 11, and 8 + 3 and 3 + 8. (We didn't put this in abstract form yet, but I hope to within the next few lessons.)

Mistakes and Misadventures

Experimentation on topics with kindergarten classes is bound to demonstrate a few outstanding things to avoid. Perhaps the most

outstanding error made in this endeavor was an attempt to present the number line in a way that played on its symetric properties. This lesson is on film.[1]

A horizontal line was drawn across the board and marked off to make a number line. The children counted spaces from zero and marked them appropriately. The number line, at that point, looked like this:

4 3 2 1 0 1 2 3 4

Then they were asked to indicate a number by standing under it. For instance, they might be asked to stand under number 3. They were brought to realize that some confusion ought to arise out of such a situation because there are doubles of each number except zero.

To remedy this confusing state of affairs, the numbers to the right of zero were called "David's numbers," and those to the left were "Terry's numbers." The appropriate initials were put by each number to indicate whose it was. The number line then looked like this:

T4 T3 T2 T1 0 D1 D2 D3 D4

Then arose the problem of general world wide communication. How does one tell all Australians what is meant by "David's 'two' "? To avoid the problem of contacting everyone in the world, they accepted the conventional notation of positive ($^+$) and negative ($^-$) numbers.

It cannot be strongly enough emphasized that this presentation, despite its surface appeal, is lethal! A close look at exactly what the children have been taught reveals that they were taught to view "negative" and "positive" only as "distinguishing numbers to the left or right of zero," and nowhere in the experience that they were given is there any indication that they were taught there should be a quantity difference—that the positive numbers are greater than the negative numbers. On the contrary, since they were taught that the numbers were measured off in equal spaces from zero, these numbers must represent quantities of the same size, only in different locations.

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[1] This material is available on the film listed as No. 79 in *U.S.O.E., 1965.* (Cf. p. 36 of the present volume.)

Another topic explored was the creation of a pseudo-isomorphism —a forced analogy between the numbers and the heights of children. A group of students were lined up according to heights and then assigned numbers, with the smallest being "zero," the next "one," etc. Then questions such as, "Which is the larger, seven or five?" were asked. The children numbered "seven" and "five" came forward and compared heights to ascertain which was larger, and then gave the answer about the numbers.

The main problem with this is that the child numbered "three," standing on the head of the child numbered "one," would not have a total height equal to that of the child numbered "four." The additive property did not hold up, and the "real" ordering property of the natural numbers did not come through.

There are other ways to set up these pseudo-isomorphisms so that the operations are better preserved. These were the two main "wrong alleys" that were explored. A few problems in the pedagogy of the topics that did work were briefly:

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1. Do not use bouquets of flowers instead of handfuls of rocks when teaching the unit on base ten. A bunch of eight flowers cannot be distinguished from a bunch of ten.

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2. In the "stones in the box" game, if you limit the number of stones the children have to work with, they will tend to keep the numbers small.

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3. When using variables, kindergarten children often cannot distinguish between $\triangle$ and $\triangledown$, so avoid using them together.

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4. The task the teacher expects of the children must be very clear to them. If they are confused about the mechanics of the task, they cannot possibly understand the material.

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5. Avoid long lessons, especially long lessons on one topic. A rapid succession of many topics can be more effective than a long lesson on one topic.

Kindergarten children are at an age where discovering the workings of their environment is their full-time job. It is our hope that the topics cited above will help them discover the workings of the number system which will eventually be one of the governing factors of their thinking.

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